HBSTM tutorial

This tutorial pretends to briefly show the nomenclature and the methodology used in the HBSTM package and make easier the visualization of the parameters structure of these kind of models. Then, we proceed to present the functions implemented in the package under two points of view:

- Beginner user: we consider that the user is nearly introduced to the HBSTM methodology and we present the basic functions.
- Expert user: we consider that the user is an expert in the HBSTM methodology and we present all the functions with all the attributes that this kind of user can exploit.

1. Introduction to the Hierarchical Bayesian Space-Time models[1]

In this section the HBSTM is briefly introduced, using notation similar to Wikle, Berliner and Cressie (1998) and Chen, Fuentes and Davis (2006), in order to introduce the notation of these kinds of models and to establish a starting point for the methodology proposed in this article.

Our interest in this case is to estimate the non-observable process $Y(s,t)$ from the observed process $Z(s,t)$, assuming that the data $Z(s,t)$ is coming from a regular grid with m spatial locations and T temporal observations for each spatial point. In other words, $s = 1, \dots, m$ and $t = 1, \dots, T$ under the assumption that the data follows a Gaussian process.

It is important to remark that the library `HBSTM` is implemented to predict regular grids. In case you want to predict an irregular grid, we recommend to develop a regular grid containing the irregular predicted points.

The latent variable $Y(s,t)$ is related with $Z(s,t)$ and is defined as:

$$Z(s,t) = K \cdot Y(s,t) + \varepsilon(s,t) \qquad (1.1)$$

Where K is an $m \times S$ matrix which relates the spatial-temporal points between Y and Z; and $\varepsilon(s,t)$ is a Gaussian error defined as multivariate white noise, assuming that all random noises are independent in space and time.

The non-observed process is defined as:

$$Y(s,t) = \mu(s) + M(s,t) + X(s,t) + \gamma(s,t) \qquad (1.2)$$

Where $\mu(s)$ represents the spatial mean, usually defined as a Markov Random Field (MRF) because of the set of positions where the field is defined in a grid and complies with the Markov property (in the sense that each observation depends strictly on the previous observation). In other words, this class of models represents the random variables and their conditional

[1] HBSTM: An R package for Hierarchical Bayesian Space-Time models, contains part of this tutorial.

dependences. Depending on the spatial structure of the data, the mean model will have, more or less, spatial dependence orders.

$M(s,t)$ is the large-scale temporal component and contains the required model seasonalities. It is usually designed to vary spatially and the temporal part is defined using trigonometric functions.

$X(s,t)$ is the small-scale temporal component and contains temporal and spatial dynamics. It is modeled using 'space-time autoregressive moving-average (STARMA) and $\gamma(s,t)$ is the random error following a Gaussian distribution.

Note that the three structures $\mu(s)$, $M(s,t)$ y $X(s,t)$ are mutually independent.

The estimation process of the HBSTM is based on a Markov chain Monte Carlo (MCMC) methodology, using a Gibbs sampling approach. For this reason, the derivation of the full conditional distributions for all the parameters of the model is needed.

As an example, we present a simple model which is shown in Wikle, Berliner and Cressie (1998). This model is defined as:

$$\vec{\boldsymbol{Y}}_t = \vec{\boldsymbol{\mu}} + \vec{\boldsymbol{M}}_t + \vec{\boldsymbol{X}}_t + \vec{\boldsymbol{\gamma}}_t \qquad \forall s \in S \qquad (1.\ 3)$$

Taking into account that for each space point s, the time representation is obtained, where $\vec{\boldsymbol{\mu}}$ is built using a nearest-neighbor model (which is defined in section 2.3) with first-order spatial dependence. $\vec{\boldsymbol{M}}_t$ has one seasonality $'w'$ and is defined using a trigonometric function. $\vec{\boldsymbol{X}}_t$ is defined by a STARMA model with one spatial and temporal lag and $\vec{\boldsymbol{\gamma}}_t$ is the error. Figure 1.1 contains the parameter structure of the model in order to facilitate the 'global vision' of the problem:

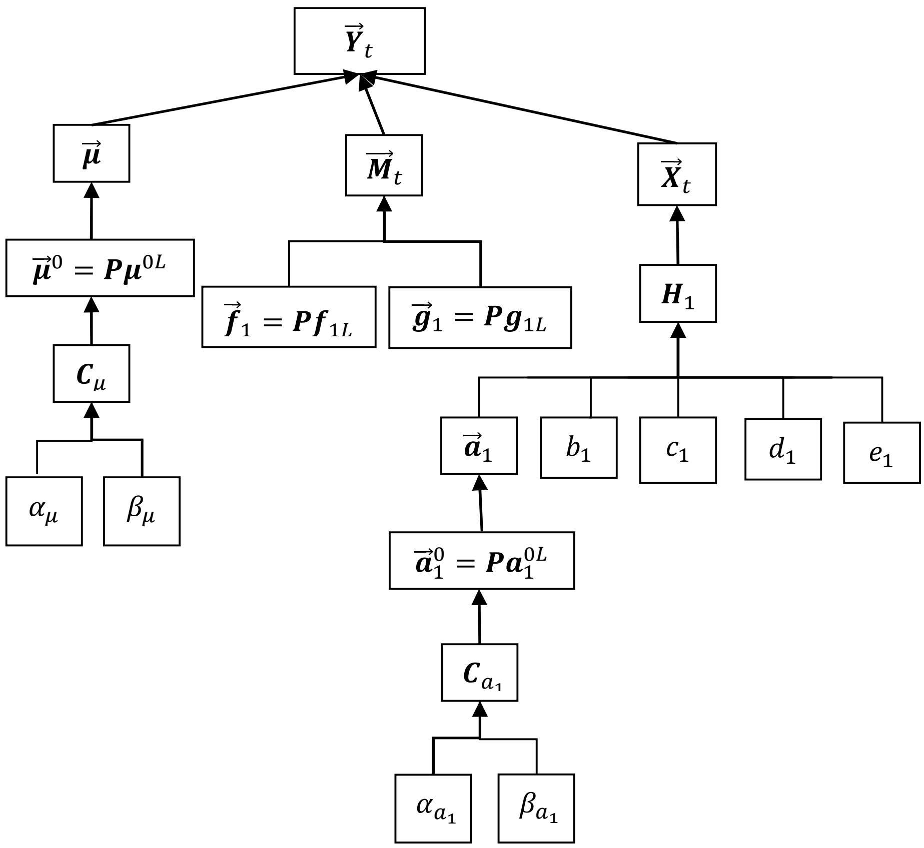


Figure 1.1: Model parameter structure

Where the description of labels in Figure 1, from bottom to top, is:

- α_μ , β_μ: Spatial relation coefficients east-west and north-south of $\vec{\boldsymbol{\mu}}$
- $\boldsymbol{C}_\mu$: Spatial relation matrix which contains the coefficients α_μ and β_μ
- $\vec{\boldsymbol{\mu}}, \vec{\boldsymbol{\mu}}^0, \boldsymbol{\mu}^{0L}$: Spatial mean
- α_{a_1} , β_{a_1}: Spatial relation coefficients east-west and north-south of $\vec{\boldsymbol{a}}_1$
- $\boldsymbol{C}_{a_1}$: Spatial relation matrix which contains the coefficients α_{a_1} and β_{a_1}
- $\vec{\boldsymbol{a}}_1, \vec{\boldsymbol{a}}_1^0, \boldsymbol{a}_1^{0L}$: Temporal Autoregressive coefficients in the same spatial point.
- b_1, c_1, d_1, e_1: Temporal autoregressive coefficients from point s with its neighbors.
- $\boldsymbol{H}_1$: Spatial-temporal relation matrix which contains the coefficients $\vec{\boldsymbol{a}}_1, b_1, c_1, d_1$ and e_1
- $\boldsymbol{P} = \left(\boldsymbol{1}\ \overrightarrow{long}\ \overrightarrow{lat}\right) \equiv S \times 3$ linear trend 'design' matrix
- $\vec{\boldsymbol{f}}, \boldsymbol{f}_{L1}$: 'cosine' coefficients
- $\vec{\boldsymbol{g}}, \boldsymbol{g}_{L1}$: 'sine' coefficients

If we analyze this example carefully, we notice that it would be one of the simplest HBSTM models, in that it contains only one seasonality, one temporal lag and the simplest of spatial structures.

As is the case in other methodologies, the complexity model increases depending on the observed process $Z(s,t)$. It could need more seasonalities and more spatial or temporal lags, thereby making the parameter structure and its estimation procedure more complex. Moreover, this complexity could make it more difficult to obtain the best model for fitting the data.

This complexity causes the space-time structure and conditional distributions to vary for each model, and therefore more difficult to implement them. Luckily, some patterns exist for extending the model and calculating the conditional distribution.

As a result, we could generalize the calculations for extending those kinds of models. An important contribution in this tutorial is that we explain how to implement the HBSTM with N seasonalities, K temporal lags and other more complex spatial structures.

2. Generic model definition

In this section we define a generic model definition of the HBSTM methodology. This model has the following characteristics:

- N seasonalities in the $\vec{\boldsymbol{M}}_t$ component.
- K autoregressive temporal lags in the $\vec{\boldsymbol{X}}_t$ component.
- The spatial structure of the model is defined as a full lagged rose diagram.

This is the general structure implemented in our proposed package and, in the following definitions we will link the theoretical parameters with its names defined in the classes and attributes of our package.

2.1. First stage: measurement process

Now, the considered model is

$$\vec{\boldsymbol{Z}}_t = \boldsymbol{K}\vec{\boldsymbol{Y}}_t + \vec{\boldsymbol{\epsilon}}_t \qquad (2.\ 1)$$

where $\vec{\boldsymbol{Z}}_t$ is an $m \times 1$ vector of observations; $\vec{\boldsymbol{Y}}_t$ is an $S \times 1$ state vector; $\boldsymbol{K}$ is an $m \times S$ matrix that maps the temperature values at grid locations according to the observations; and $\vec{\boldsymbol{\epsilon}}_t$ is an $m \times 1$ vector representing the error. We assume that $\vec{\boldsymbol{Z}}_t$ is conditionally Gaussian:

$$\left[\vec{\boldsymbol{Z}}_t | \boldsymbol{K}, \vec{\boldsymbol{Y}}_t, \sigma_\epsilon^2\right] \sim Gau\left(\boldsymbol{K}\vec{\boldsymbol{Y}}_t, \sigma_\epsilon^2 \boldsymbol{I}\right) \qquad (2.\ 2)$$

2.2. Second stage: large and small scale features

At this point, the model for the state process $\vec{\boldsymbol{Y}}_t$ is

$$\vec{\boldsymbol{Y}}_t = \vec{\boldsymbol{\mu}} + \vec{\boldsymbol{M}}_t + \vec{\boldsymbol{X}}_t + \vec{\gamma}_t \qquad (2.\ 3)$$

where $\vec{\boldsymbol{\mu}}$ are the spatial means defined with a spatial structure; $\vec{\boldsymbol{M}}_t$ is the spatial seasonal component; $\vec{\boldsymbol{X}}_t$ is the intra-seasonal space-time component; and $\vec{\boldsymbol{\gamma}}_t$ is an error term. We assume that $\vec{\boldsymbol{Y}}_t$ is conditionally Gaussian:

$$\left[\vec{\boldsymbol{Y}}_t | \vec{\boldsymbol{\mu}}, \vec{\boldsymbol{M}}_t, \vec{\boldsymbol{X}}_t, \vec{\boldsymbol{\gamma}}_t\right] \sim Gau\left(\vec{\boldsymbol{\mu}} + \vec{\boldsymbol{M}}_t + \vec{\boldsymbol{X}}_t, \sigma_\gamma^2 \boldsymbol{I}\right) \qquad (2.\ 4)$$

For further explanation about this part, see section A.2 in Wikle, Berliner and Cressie (1998) and López and Muñoz (2013)

2.3. Third stage: spatial structures

We assume that $\vec{\boldsymbol{\mu}}$, $\vec{\boldsymbol{M}}_t$ and $\vec{\boldsymbol{X}}_t$ are mutually independent, conditional on third-stage parameters.

$\vec{\boldsymbol{\mu}}$ is defined using a Gaussian MRF (Markov Random Field) with full-lagged rose diagram spatial dependence (see plot…)

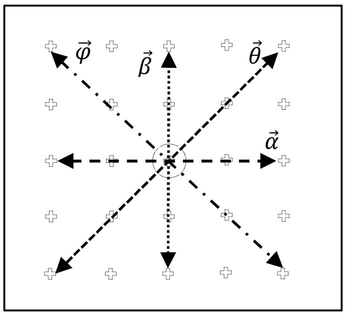


Figure 2.1: Full-lagged rose diagram structure

Where $\vec{\alpha}$, $\vec{\beta}$, $\vec{\varphi}$ and $\vec{\theta}$ have the same pattern in both senses of the same direction:

- $\vec{\alpha} = \{\alpha^1, \dots, \alpha^{n_1}\}$
- $\vec{\beta} = \{\beta^1, \dots, \beta^{n_2}\}$
- $\vec{\varphi} = \{\varphi^1, \dots, \varphi^{n_3}\}$
- $\vec{\theta} = \{\theta^1, \dots, \theta^{n_4}\}$

That is, for grid locations $k = 1, \dots, m$ and $l = 1, \dots, n$, where k and l are indices correspondig to the longitudes and the latitudes, the conditional distribution of $\mu(k,l)$ is:

$$\mu(k,l)|\{\mu(i,j):(i,j) \neq (k,l)\} \sim Gau\left\{\mu^0(k,l) + \sum_{i\in n_1} \alpha_\mu^i\{(\mu(k-i,l)-\mu^0(k-i,l)) + (\mu(k+i,l)-\mu^0(k+i,l))\} + \sum_{i\in n_2} \beta_\mu^i\{(\mu(k,l-i)-\mu^0(k,l-i)) + (\mu(k,l+i)-\mu^0(k,l+i))\} + \sum_{i\in n_3} \varphi_\mu^i\{(\mu(k-i,l-i)-\mu^0(k-i,l-i)) + (\mu(k+i,l+i)-\mu^0(k+i,l+i))\} + \sum_{i\in n_4} \theta_\mu^i\{(\mu(k-i,l+i)-\mu^0(k-i,l+i)) + (\mu(k+i,l-i)-\mu^0(k+i,l-i))\}, \tau_\mu^2\right\}$$

(2. 5)

Where $\vec{\boldsymbol{\mu}}^0$ is the gridpoint specific MRF mean, $\vec{\boldsymbol{\alpha}}_\mu$, $\vec{\boldsymbol{\beta}}_\mu$, $\vec{\boldsymbol{\varphi}}_\mu$ and $\vec{\boldsymbol{\theta}}_\mu$(east-weast, north-south, northweast-southeast and northeast-southweast, respectively) MRF spatial despendence parámeters and τ_μ^2 is the variance. The mean $\vec{\boldsymbol{\mu}}$ also can be defined as:

$$\vec{\boldsymbol{\mu}}|\{\vec{\boldsymbol{\mu}}^0, \tau_\mu^2, \vec{\boldsymbol{\alpha}}_\mu, \vec{\boldsymbol{\beta}}_\mu, \vec{\boldsymbol{\varphi}}_\mu, \vec{\boldsymbol{\theta}}_\mu\} \sim Gau\left(\vec{\boldsymbol{\mu}}^0, \left(\boldsymbol{I} - \boldsymbol{C}_\mu\right)^{-1}\tau_\mu^2\right) \quad (2.\ 6)$$

Where $\boldsymbol{C}_\mu$ is an $S \times S$ matrix with the off-diagonals given by $\vec{\boldsymbol{\alpha}}_\mu$, $\vec{\boldsymbol{\beta}}_\mu$, $\vec{\boldsymbol{\varphi}}_\mu$ and $\vec{\boldsymbol{\theta}}_\mu$.

The seasonal component is defined with N seasonalities with amplitudes and phases that vary spatially:

$$\vec{\boldsymbol{M}}_t = \sum_{i=1}^{N} \vec{\boldsymbol{f}}_i \cos(w_i t) + \vec{\boldsymbol{g}}_i \sin(w_i t)$$

(2. 7)

Where:

- w_i = the seasonality which corresponds to $\frac{2\pi}{k_i}$, where k_i is the period of the seasonality $i = 1, \dots, N$.
- N = total number of seasonalities

Moreover, $\vec{\boldsymbol{f}}_i$ and $\vec{\boldsymbol{g}}_i$ $i = 1, \dots, N$ are spatially varying 'cosine' and 'sine' coefficients respectively, and are defined for $i = 1, \dots, N$:

$$f_i(k,l) = f_i[1] + f_i[2]long(k) + f_i[3]lat((k) \qquad (2.\ 7)$$

$$g_i(k,l) = g_i[1] + g_i[2]long(k) + g_i[3]lat((k) \qquad (2.\ 8)$$

Where $f_i[j]$ and $g_i[j]$, $j = 1,2,3$ are independent Gaussian random variables:

$$f_i[j] \sim Gau\left(\tilde{f}_i[j], \tilde{\sigma}^2_{f_i}[j]\right) \qquad (2.\ 9)$$

$$g_i[j] \sim Gau\left(\tilde{g}_i[j], \tilde{\sigma}^2_{g_i}[j]\right) \qquad (2.\ 10)$$

and their parameters are fixed and specified. Then, generalizing,

$$\vec{\boldsymbol{f}}_i = \left(\mathbf{1}\ \overrightarrow{long}\ \overrightarrow{lat}\right)(f_i[1], f_i[2], f_i[3])' = \boldsymbol{P}\boldsymbol{f}_{iL} \qquad (2.\ 11)$$

Where $\boldsymbol{P}$ is a $S \times 3$ linear trend 'design' matrix and $\boldsymbol{f}_{iL} \equiv (f_i[1], f_i[2], f_i[3])'$. Then, from their independence

$$\boldsymbol{f}_{iL} | \tilde{\boldsymbol{f}}_{iL}, \widetilde{\boldsymbol{\Sigma}}_{f_i} \sim Gau\left(\tilde{\boldsymbol{f}}_{iL}, \widetilde{\boldsymbol{\Sigma}}_{f_i}\right) \qquad (2.\ 12)$$

where $\tilde{\boldsymbol{f}}_{iL} \equiv \left(\tilde{f}_i[1], \tilde{f}_i[2], \tilde{f}_i[3]\right)'$ and $\widetilde{\boldsymbol{\Sigma}}_{f_i}$ is a 3 by 3 diagonal matrix with $\tilde{\sigma}^2_{f_i}[1]$, $\tilde{\sigma}^2_{f_i}[2]$, and $\tilde{\sigma}^2_{f_i}[3]$ variances on the main diagonal.

Similarly,

$$\vec{\boldsymbol{g}}_i = \left(\mathbf{1}\ \overrightarrow{long}\ \overrightarrow{lat}\right)(g_i[1], g_i[2], g_i[3])' = \boldsymbol{P}\boldsymbol{g}_{iL} \qquad (2.\ 13)$$

where $\boldsymbol{g}_{iL} \equiv (g_i[1], g_i[2], g_i[3])'$. Then, from their independence

$$\boldsymbol{g}_{iL} | \tilde{\boldsymbol{g}}_{iL}, \widetilde{\boldsymbol{\Sigma}}_{g_i} \sim Gau\left(\tilde{\boldsymbol{g}}_{iL}, \widetilde{\boldsymbol{\Sigma}}_{g_i}\right) \qquad (2.\ 14)$$

where $\tilde{\boldsymbol{g}}_{iL} \equiv (\tilde{g}_i[1], \tilde{g}_i[2], \tilde{g}_i[3])'$ and $\widetilde{\boldsymbol{\Sigma}}_{g_i}$ is a 3 by 3 diagonal matrix with $\tilde{\sigma}^2_{g_i}[1]$, $\tilde{\sigma}^2_{g_i}[2]$, and $\tilde{\sigma}^2_{g_i}[3]$ variances on the main diagonal.

In the general case, the small-scale space-time term is modeled as a VAR model process.

$$\vec{\boldsymbol{X}}_t = \sum_{r=1}^{K} \boldsymbol{H}_r \vec{\boldsymbol{X}}_{t-r} + \vec{\boldsymbol{\eta}}_t \qquad \forall s \in S$$

(2. 16)

Where:

- $K =$ number of lags
- $\vec{\boldsymbol{\eta}}_t$ is the random error following a Gaussian distribution
- $\boldsymbol{H}_r$ is a $S \times S$ space-time matrix

The spatial relationships between the temporal lags are stored in the $\boldsymbol{H}_r$ matrix, although, depending on this defined spatial relation, the distribution of the parameters in $\boldsymbol{H}_r$ may change.

Regarding the parameters in $\boldsymbol{H}_r$, the vector $\vec{\boldsymbol{a}}_r$ is located on the main diagonal and it is the 'pure' autoregressive parameter, i. e., it is the relation of a spatial point with itself in the past. In the subdiagonals are stored the temporal autoregressive parameters of a spatial point related with its spatial neighbors. In other words, and using the example in Figure 2.2 of the fixed temporal lag r and spatial point 1, $h_r^{1,1} X_{t-r}^1$ shows the temporal relationship of spatial point 1 with itself while $\sum_{i=2}^{S} h_r^{1,i} X_{t-r}^i$ exhibits the temporal relationship with the rest of the spatial points of the grid.

$$\boldsymbol{H}_r \vec{\boldsymbol{X}}_{t-r} = \begin{pmatrix} h_r^{1,1} & h_r^{1,2} & \dots & h_r^{1,S} \\ \vdots & \vdots & \ddots & \vdots \\ & & \dots & \end{pmatrix} \begin{pmatrix} X_{t-r}^1 \\ \vdots \\ X_{t-r}^S \end{pmatrix}$$

Figure 2.2: $\boldsymbol{H}_r$ and $\vec{\boldsymbol{X}}_{t-r}$ multiplication

We consider the full-lagged rose diagram spatial dependence VAR model (Figure 2. 3) and the vectors which contains that special structure are defined as:

- $\vec{\mathbf{W}}_r = Weast = \{W_r^1, \dots, W_r^{n_1}\}$
- $\vec{\mathbf{E}}_r = East = \{E_r^1, \dots, E_r^{n_1}\}$
- $\vec{\mathbf{N}}_r = North = \{N_r^1, \dots, N_r^{n_2}\}$
- $\vec{\mathbf{S}}_r = South = \{S_r^1, \dots, S_r^{n_2}\}$
- $\overrightarrow{\mathbf{SE}}_r = SouthEast = \{SE_r^1, \dots, SE_r^{n_3}\}$
- $\overrightarrow{\mathbf{NW}}_r = NorthWeast = \{NW_r^1, \dots, NW_r^{n_3}\}$
- $\overrightarrow{\mathbf{SW}}_r = SouthWeast = \{SW_r^1, \dots, SW_r^{n_4}\}$
- $\overrightarrow{\mathbf{NE}}_r = NorthEast = \{NE_r^1, \dots, NE_r^{n_4}\}$

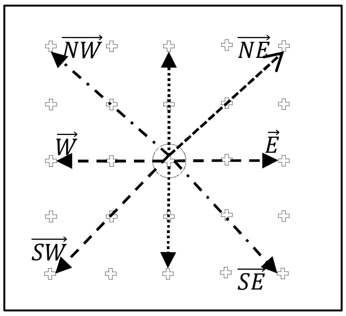


Figure 2.3: Full-lagged rose diagram structure considering different behaviors in both ways of one direction

Thus, we define $X\big((k,l),t\big)$ as:

$$
\begin{aligned}
X\big((k,l),t\big) = \sum_{r\in\{1,\dots,K\}} \Bigg\{ & a_r(k,l)X\big((k,l),t-i\big) \\
& + \sum_{j=1}^{n_1}\left[E_r^j X\big((k,l+j),t-i\big) + W_r^j X\big((k,l-j),t-i\big)\right] \\
& + \sum_{j=1}^{n_2}\left[N_r^j X\big((k+j,l),t-i\big) + S_r^j X\big((k-j,l),t-i\big)\right] \\
& + \sum_{j=1}^{n_3}\left[SE_r^j X\big((k+j,l+j),t-i\big) + NW_r^j X\big((k-j,l-j),t-i\big)\right] \\
& + \sum_{j=1}^{n_4}\left[SW_r^j X\big((k+j,l-j),t-i\big) + NE_r^j X\big((k-j,l+j),t-i\big)\right]\Bigg\} \\
& + \eta\big((k,l),t\big)
\end{aligned}
$$

(2. 15)

And we assume

$$\vec{\boldsymbol{X}}_t|\{\vec{\boldsymbol{X}}_{t-1},\boldsymbol{H}_1,\dots,\vec{\boldsymbol{X}}_{t-K},\boldsymbol{H}_K,\sigma_n^2\}\sim Gau\big(\boldsymbol{H}_1\vec{\boldsymbol{X}}_{t-1}+\cdots+\boldsymbol{H}_K\vec{\boldsymbol{X}}_{t-K},\sigma_\eta^2\boldsymbol{I}\big) \quad (2.\ 16)$$

Where $\boldsymbol{H}_r$ is a diagonal matrix with off-diagonals given by $\overrightarrow{\mathbf{W}}_r$, $\vec{\mathbf{E}}_r$, $\vec{\mathbf{N}}_r$, $\vec{\mathbf{S}}_r$, $\overrightarrow{\mathbf{SE}}_r$, $\overrightarrow{\mathbf{NW}}_r$, $\overrightarrow{\mathbf{SW}}_r$ and $\overrightarrow{\mathbf{NE}}_r$ directions and the main diagonal contains the vector $\vec{\boldsymbol{a}}_r$

2.4. Fourth stage: priors on parameters

The mean $\vec{\boldsymbol{\mu}}^0$ is represented as:

$$\mu_0(k,l) = \mu_0[1] + \mu_0[2]long(k) + \mu_0[3]lat(l) \qquad (2.\ 17)$$

Where $long(k)$ and $lat(l)$ are the longitude and latitude of the (k,l)th gridpoint, respectively and the regression coefficients $\mu_0[1]$, $\mu_0[2]$, $\mu_0[3]$ are specified to be independent Gaussian random variables:

$$\mu_0[j]\sim Gau\big(\tilde{\mu}_0[j],\tilde{\sigma}_{\mu_0}^2[j]\big) \qquad j=1,2,3 \qquad (2.\ 18)$$

Where their parameters are fixed and specified.

Then, generalizing,

$$\vec{\boldsymbol{\mu}}^0 = \left(\mathbf{1}\ \overrightarrow{long}\ \overrightarrow{lat}\right)(\mu^0[1], \mu^0[2], \mu^0[3])' = \boldsymbol{P}\boldsymbol{\mu}^{0L} \qquad (2.\ 19)$$

Where $\boldsymbol{P}$ is a $S \times 3$ linear trend 'design' matrix, and $\boldsymbol{\mu}^{0L} \equiv (\mu^0[1], \mu^0[2], \mu^0[3])'$ is the linear trend parameter vector. Then, from their independence

$$\boldsymbol{\mu}^{0L} \sim Gau\left(\tilde{\boldsymbol{\mu}}^{0L}, \tilde{\boldsymbol{\Sigma}}_{\mu^{0L}}\right) \qquad (2.\ 20)$$

Where $\tilde{\boldsymbol{\Sigma}}_{\mu^{0L}}$ is a 3 x 3 diagonal matrix with $\tilde{\sigma}^2_{\mu^{0L}}[1]$, $\tilde{\sigma}^2_{\mu^{0L}}[2]$ and $\tilde{\sigma}^2_{\mu^{0L}}[3]$ variances on the main diagonal.

The vectors $\vec{\boldsymbol{\alpha}}_\mu$, $\vec{\boldsymbol{\beta}}_\mu$, $\vec{\boldsymbol{\varphi}}_\mu$ and $\vec{\boldsymbol{\theta}}_\mu$ are defined as independent Gaussian random parameters, but constrained to ensure positive-definieness of C_μ

$$\alpha^i_\mu \sim Gau\left(\tilde{\alpha}^i_\mu, \tilde{\sigma}^2_{\alpha^i_\mu}\right) \quad i = 1, \dots, n_1 \qquad (2.\ 21)$$

$$\beta^i_\mu \sim Gau\left(\tilde{\beta}^i_\mu, \tilde{\sigma}^2_{\beta^i_\mu}\right) \quad i = 1, \dots, n_2 \qquad (2.\ 22)$$

$$\varphi^i_\mu \sim Gau\left(\tilde{\varphi}^i_\mu, \tilde{\sigma}^2_{\varphi^i_\mu}\right) \quad i = 1, \dots, n_3 \qquad (2.\ 23)$$

$$\theta^i_\mu \sim Gau\left(\tilde{\theta}^i_\mu, \tilde{\sigma}^2_{\theta^i_\mu}\right) \quad i = 1, \dots, n_4 \qquad (2.\ 24)$$

Where their parameters are fixed and specified.

We define de autoregressive parameters $\vec{\boldsymbol{a}}_r$ $(r = 1, \dots, K)$ using a Gaussian MRF with full-lagged rose diagram spatial dependence (Figure 2. 1) in order to fit our proposed model to the case of three seasonalities

$$\begin{aligned} a_r(k,l) | \{a_r(i,j) &: (i,j) \\ &\neq (k,l)\} \sim Gau\Bigg\{ a^0_r(k,l) \\ &+ \sum_{i \in n_1} \alpha^i_{a_r} \left\{ \left(a_r(k-i,l) - a^0_r(k-i,l)\right) + \left(a_r(k+i,l) - a^0_r(k+i,l)\right) \right\} \\ &+ \sum_{i \in n_2} \beta^i_{a_r} \left\{ \left(a_r(k,l-i) - a^0_r(k,l-i)\right) + \left(a_r(k,l+i) - a^0_r(k,l+i)\right) \right\} \\ &+ \sum_{i \in n_3} \varphi^i_{a_r} \left\{ \left(a_r(k-i,l-i) - a^0_r(k-i,l-i)\right) \right. \\ &\left. + \left(a_r(k+i,l+i) - a^0_r(k+i,l) + i\right) \right\} \\ &+ \sum_{i \in n_4} \theta^i_{a_r} \left\{ \left(a_r(k-i,l+i) - a^0_r(k-i,l+i)\right) \right. \\ &\left. + \left(a_r(k+i,l-i) - a^0_r(k+i,l-i)\right) \right\}, \tau^2_{a_r} \Bigg\} \end{aligned}$$

(2. 25)

Where $\vec{\boldsymbol{a}}_r^0$ is the gridpoint specific MRF mean, $\vec{\boldsymbol{\alpha}}_{a_r}$, $\vec{\boldsymbol{\beta}}_{a_r}$, $\vec{\boldsymbol{\varphi}}_{a_r}$ and $\vec{\boldsymbol{\theta}}_{a_r}$(east-weast, north-south, northweast-southeast and northeast-southweast vectors, respectively) MRF spatial despendence parameters and $\tau_{a_r}^2$ is the variance.

The parameters $\vec{\boldsymbol{\alpha}}_\mu$, $\vec{\boldsymbol{\beta}}_\mu$, $\vec{\boldsymbol{\varphi}}_\mu$ and $\vec{\boldsymbol{\theta}}_\mu$ are stored in the off-diagonals of the $S \times S$ matrix $\boldsymbol{C}_{a_r}$.

The autoregressive parameters, that represent the full-lagged rose diagram spatial dependence, are assumed that follow independent Gaussian distributions.

$$W_r^i \sim Gau(\widetilde{W}_{0r}^i, \tilde{\sigma}_{W_r^i}^2) \quad i = 1, \dots, n_1 \qquad (2.\ 26)$$

$$E_r^i \sim Gau(\tilde{E}_{0r}^i, \tilde{\sigma}_{E_r^i}^2) \quad i = 1, \dots, n_1 \qquad (2.\ 27)$$

$$N_r^i \sim Gau(\widetilde{N}_{0r}^i, \tilde{\sigma}_{N_r^i}^2) \quad i = 1, \dots, n_2 \qquad (2.\ 28)$$

$$S_r^i \sim Gau(\tilde{S}_{0r}^i, \tilde{\sigma}_{S_r^i}^2) \quad i = 1, \dots, n_2 \qquad (2.\ 29)$$

$$\mathrm{SE}_r^i \sim Gau(\widetilde{SE}_{0r}^i, \tilde{\sigma}_{SE_r^i}^2) \quad i = 1, \dots, n_3 \qquad (2.\ 30)$$

$$NW_r^i \sim Gau(\widetilde{NW}_{0r}^i, \tilde{\sigma}_{NW_r^i}^2) \quad i = 1, \dots, n_3 \qquad (2.\ 31)$$

$$\mathrm{SW}_r^i \sim Gau(\widetilde{SW}_{0r}^i, \tilde{\sigma}_{SW_r^i}^2) \quad i = 1, \dots, n_4 \qquad (2.\ 32)$$

$$NE_r^i \sim Gau(\widetilde{NE}_{0r}^i, \tilde{\sigma}_{NE_r^i}^2) \quad i = 1, \dots, n_4 \qquad (2.\ 33)$$

Where their parameters are fixed and specified.

For the variances specified in stages one though three, independence between them is assumed and use the conjugate priors

$$\sigma_\varepsilon^2 \sim IG(\tilde{q}_\varepsilon, \tilde{r}_\varepsilon) \qquad (2.\ 34)$$

$$\sigma_\gamma^2 \sim IG\left(\tilde{q}_\gamma, \tilde{r}_\gamma\right) \qquad (2.\ 35)$$

$$\sigma_\eta^2 \sim IG\left(\tilde{q}_\eta, \tilde{r}_\eta\right) \qquad (2.\ 36)$$

$$\tau_\mu^2 \sim IG\left(\tilde{q}_\mu, \tilde{r}_\mu\right) \qquad (2.\ 37)$$

2.5. Fifth stage: hyperpriors

Assumed that $\vec{\boldsymbol{a}}_{0r}$ has simple spatial trend structure:

$$a_r^0(k, l) = a_r^0[1] + a_r^0[2]long(k) + a_r^0[3]lat(l) \qquad r \in \{1, \dots, K\} \qquad (2.\ 38)$$

Where the regression coefficients $a_r^0[1]$, $a_r^0[2]$, $a_r^0[3]$ are specified to be independent Gaussian random variables:

$$a_r^0[j] \sim Gau\left(\tilde{a}_r^0[j], \tilde{\sigma}_{a_r^0}^2[j]\right) \qquad j = 1,2,3 \qquad (2.\ 39)$$

and their parameters are fixed and specified. Then, generalizing (2. 40),

$$\vec{\boldsymbol{a}}_1^0 = \left(\mathbf{1}\ \overrightarrow{long}\ \overrightarrow{lat}\right)(a_1^0[1], a_1^0[2], a_1^0[3])' = \boldsymbol{P}\boldsymbol{a}_1^{0L} \qquad (2.40)$$

Where $\boldsymbol{P}$ is a $S \times 3$ linear trend 'design' matrix, and $\boldsymbol{a}_1^{0L} \equiv (a_1^0[1], a_1^0[2], a_1^0[3])'$ is the linear trend parameter vector. Then, from their independence

$$\boldsymbol{a}_1^{0L} \sim Gau\left(\tilde{\boldsymbol{a}}_1^{0L}, \widetilde{\boldsymbol{\Sigma}}_{a_1^{0L}}\right) \qquad (2.41)$$

Where $\widetilde{\boldsymbol{\Sigma}}_{a_1^{0L}}$ is a 3 x 3 diagonal matrix with $\tilde{\sigma}^2_{a_1^{0L}}[1]$, $\tilde{\sigma}^2_{a_1^{0L}}[2]$ and $\tilde{\sigma}^2_{a_1^{0L}}[3]$ on the main diagonal.

We define $\vec{\boldsymbol{\alpha}}_{a_r}$, $\vec{\boldsymbol{\beta}}_{a_r}$, $\vec{\boldsymbol{\varphi}}_{a_r}$ and $\vec{\boldsymbol{\theta}}_{a_r}$ as independent Gaussian random variables, but constrained to ensure positive-definieness of C_{a_r}

$$\alpha^i_{a_r} \sim Gau\left(\tilde{\alpha}^i_{a_r}, \tilde{\sigma}^2_{\alpha^i_{a_r}}\right) \quad i = 1, \dots, n_1 \qquad (2.42)$$

$$\beta^i_{a_r} \sim Gau\left(\tilde{\beta}^i_{a_r}, \tilde{\sigma}^2_{\beta^i_{a_r}}\right) \quad i = 1, \dots, n_2 \qquad (2.43)$$

$$\varphi^i_{a_r} \sim Gau\left(\tilde{\varphi}^i_{a_r}, \tilde{\sigma}^2_{\varphi^i_{a_r}}\right) \quad i = 1, \dots, n_3 \qquad (2.44)$$

$$\theta^i_{a_r} \sim Gau\left(\tilde{\theta}^i_{a_r}, \tilde{\sigma}^2_{\theta^i_{a_r}}\right) \quad i = 1, \dots, n_4 \qquad (2.45)$$

And we use de inverse Gamma distribution to conjugate the prior $\tau^2_{a_r}$

$$\tau^2_{a_r} \sim IG\left(\tilde{q}_{a_r}, \tilde{r}_{a_r}\right) \qquad (2.46)$$

where their parameters $\tilde{q}_{a_r}$ and $\tilde{r}_{a_r}$ are fixed and specified.

2.6. Structure diagram of the model parameters

In the Figure 2.4 we show the generalized model structure to make easier to understand how is structured the object of class `HBSTM`.

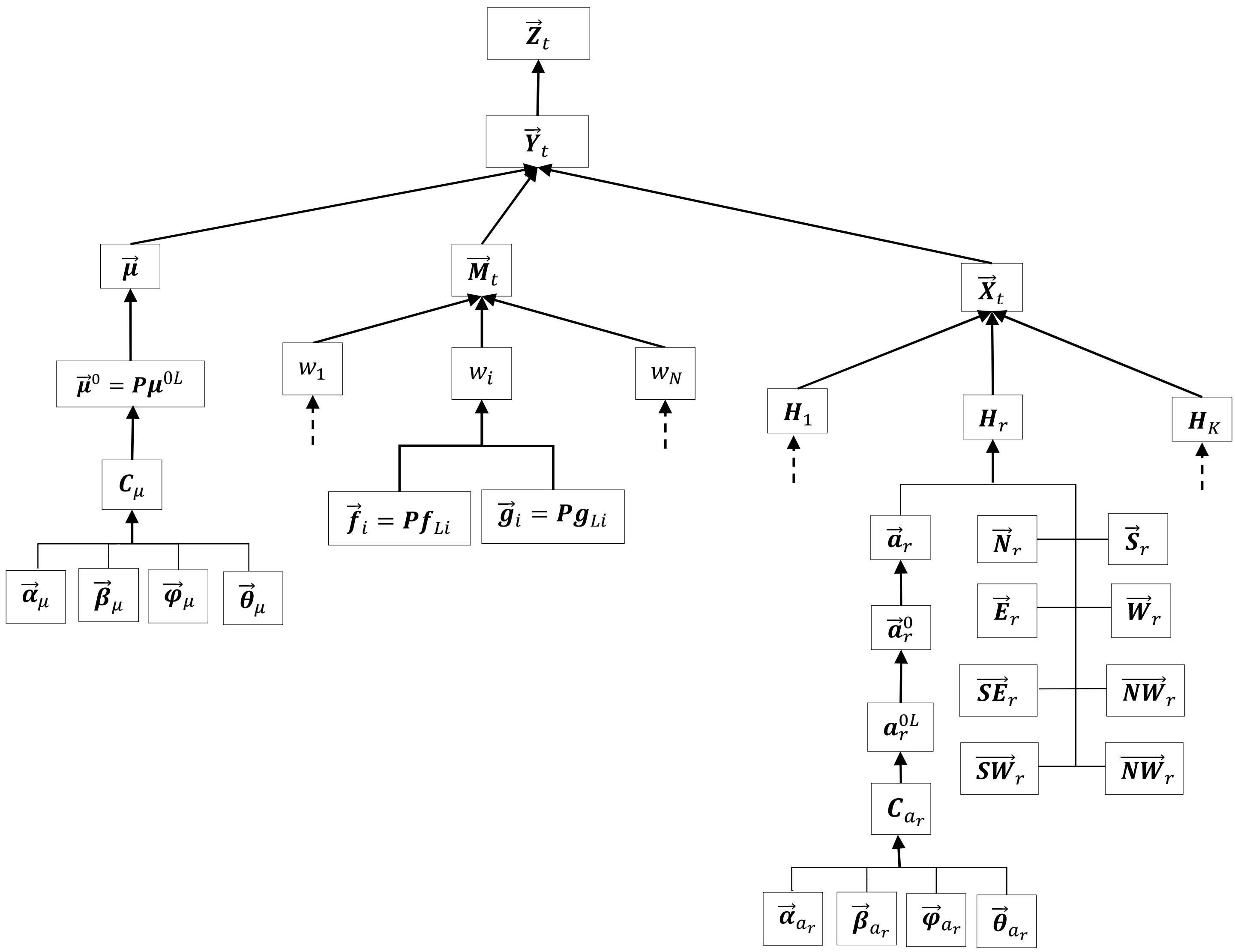


Figure 2.4: Generalized model parameters structure

3. Transcription to the model parameters

In this section we show how can be extracted all the model parameters and its hyperpriors.

Parameter	R extraction
$\vec{\boldsymbol{Z}}_t$	`Object['Zt']`
$\boldsymbol{K}$	`Object['K']`
$\boldsymbol{P}$	`Object['P']`
$\vec{\boldsymbol{Y}}_t$	`Object['Parameters']['Yt']`
σ_ϵ^2	`Object['Parameters']['sigma2E']`
$\vec{\boldsymbol{\mu}}$ component	`Object['Parameters']['Mu']`
$\vec{\boldsymbol{M}}_t$ component	`Object['Parameters']['Mt']`
$\vec{\boldsymbol{X}}_t$ component	`Object['Parameters']['Xt']`
σ_Y^2	`Object['Parameters']['sigma2Y']`
$\vec{\boldsymbol{\mu}}$	`Object['Parameters']['Mu']['muvect']`
$\vec{\boldsymbol{\mu}}^0$	`Object['Parameters']['Mu']['mu0vect']`
$\boldsymbol{\mu}^{0L}$	`Object['Parameters']['Mu']['mu0L']`
τ_μ^2	`Object['Parameters']['Mu']['sigma2Mu']`
$\boldsymbol{C}_\mu$	`Object['Parameters']['Mu']['spatialMu']['Cmat']`
$\vec{\boldsymbol{\alpha}}_\mu$	`Object['Parameters']['Mu']['spatialMu']['alpha']`
$\vec{\boldsymbol{\beta}}_\mu$	`Object['Parameters']['Mu']['spatialMu']['beta']`
$\vec{\boldsymbol{\varphi}}_\mu$	`Object['Parameters']['Mu']['spatialMu']['phi']`
$\vec{\boldsymbol{\theta}}_\mu$	`Object['Parameters']['Mu']['spatialMu']['theta']`
$\vec{\boldsymbol{M}}_t$	`Object['Parameters']['Mt']['Mt']`
w_i	`Object['Parameters']['Mt']['seas'][[i]]['w']`
$\vec{\boldsymbol{f}}_i$	`Object['Parameters']['Mt']['seas'][[i]]['fvect']`
$\boldsymbol{f}_{iL}$	`Object['Parameters']['Mt']['seas'][[i]]['f0L']`
$\vec{\boldsymbol{g}}_i$	`Object['Parameters']['Mt']['seas'][[i]]['gvect']`
$\boldsymbol{g}_{iL}$	`Object['Parameters']['Mt']['seas'][[i]]['g0L']`
$\vec{\boldsymbol{X}}_t$	`Object['Parameters']['Xt']['Xt']`
$\vec{\boldsymbol{X}}_0$	`Object['Parameters']['Xt']['X0']`
σ_η^2	`Object['Parameters']['Xt']['sigma2N']`
$\boldsymbol{H}_r$	`Object['Parameters']['Xt']['AR'][[r]]['H']`
$\vec{\boldsymbol{a}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['avect']`
$\vec{\boldsymbol{a}}_r^0$	`Object['Parameters']['Xt']['AR'][[r]]['a0vect']`
$\boldsymbol{a}_r^{0L}$	`Object['Parameters']['Xt']['AR'][[r]]['a0L']`
$\tau_{a_r}^2$	`Object['Parameters']['Xt']['AR'][[r]]['sigma2A']`
$\boldsymbol{C}_{a_r}$	`Object['Parameters']['Xt']['AR'][[r]]['spatialA']['Cmat']`
$\vec{\boldsymbol{\alpha}}_{a_r}$	`Object['Parameters']['Xt']['AR'][[r]]['spatialA']['alpha']`
$\vec{\boldsymbol{\beta}}_{a_r}$	`Object['Parameters']['Xt']['AR'][[r]]['spatialA']['beta']`
$\vec{\boldsymbol{\varphi}}_{a_r}$	`Object['Parameters']['Xt']['AR'][[r]]['spatialA']['phi']`
$\vec{\boldsymbol{\theta}}_{a_r}$	`Object['Parameters']['Xt']['AR'][[r]]['spatialA']['theta']`
$\vec{\mathbf{W}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['subdiag']['west']`
$\vec{\mathbf{E}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['subdiag']['east']`
$\vec{\mathbf{N}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['subdiag']['north']`
$\vec{\mathbf{S}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['subdiag']['south']`
$\overrightarrow{\mathbf{SE}}_r$	`Object['Parameters']['Xt']['AR'][[r]]['subdiag']['southeast']`

$\overrightarrow{\mathbf{NW}}_r$	Object[‘Parameters’][‘Xt’][‘AR’][[r]][‘subdiag’][‘northwest’]
$\overrightarrow{\mathbf{SW}}_r$	Object[‘Parameters’][‘Xt’][‘AR’][[i]][‘subdiag’][‘southwest’]
$\overrightarrow{\mathbf{NE}}_r$	Object[‘Parameters’][‘Xt’][‘AR’][[i]][‘subdiag’][‘northeast’]

Figure 3.1: Parameters model transcription to an object of class `HBSTM`

Hiperprior	R extraction
$\tilde{q}_{\varepsilon}$	Object[‘Hyperpriors’][‘ sigma2E0’][1]
$\tilde{r}_{\varepsilon}$	Object[‘Hyperpriors’][‘ sigma2E0’][2]
$\tilde{q}_{\gamma}$	Object[‘Hyperpriors’][‘ sigma2Y0’][1]
$\tilde{r}_{\gamma}$	Object[‘Hyperpriors’][‘ sigma2Y0’][2]
$\tilde{\boldsymbol{\mu}}^{0L}$	Object[‘Hyperpriors’][‘Mu0’][‘ mu0L0’]
$\tilde{\boldsymbol{\Sigma}}_{\mu^{0L}}$	Object[‘Hyperpriors’][‘Mu0’][‘ sigmu0L0’]
$\tilde{q}_{\mu}$	Object[‘Hyperpriors’][‘Mu0’][‘ sigma2Mu0’][1]
$\tilde{r}_{\mu}$	Object[‘Hyperpriors’][‘Mu0’][‘ sigma2Mu0’][2]
$\tilde{\alpha}_{\mu}^{i}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ alpha0’][1,]
$\tilde{\sigma}_{\alpha_{\mu}^{i}}^{2}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ alpha0’][2,]
$\tilde{\beta}_{\mu}^{i}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ beta0’][1,]
$\tilde{\sigma}_{\beta_{\mu}^{i}}^{2}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ beta0’][2,]
$\tilde{\varphi}_{\mu}^{i}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ phi0’][1,]
$\tilde{\sigma}_{\varphi_{\mu}^{i}}^{2}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ phi0’][2,]
$\tilde{\theta}_{\mu}^{i}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ theta0’][1,]
$\tilde{\sigma}_{\theta_{\mu}^{i}}^{2}$	Object[‘Hyperpriors’][‘Mu0’][‘ spatialMu0’][‘ theta0’][2,]
$\tilde{q}_{\eta}$	Object[‘Hyperpriors’][‘Xt0’][‘ sigma2N0’][1]
$\tilde{r}_{\eta}$	Object[‘Hyperpriors’][‘Xt0’][‘ sigma2N0’][2]
$\vec{\boldsymbol{\mu}}_{X_0}$	Object[‘Hyperpriors’][‘Xt0’][‘ X00’]
Σ_{X_0}	Object[‘Hyperpriors’][‘Xt0’][‘ sigma2X00’]
$\tilde{\boldsymbol{a}}_{r}^{0L}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ a0L0’]
$\tilde{\boldsymbol{\Sigma}}_{a_r^{0L}}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ siga0L0’]
$\tilde{q}_{a_r}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ sigma2A0’][1]
$\tilde{r}_{a_r}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ sigma2A0’][2]
$\tilde{\alpha}_{a_r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ alpha0’][1,]
$\tilde{\sigma}_{\alpha_{a_r}^{i}}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ alpha0’][2,]
$\tilde{\beta}_{a_r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ beta0’][1,]
$\tilde{\sigma}_{\beta_{a_r}^{i}}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ beta0’][2,]
$\tilde{\varphi}_{a_r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ phi0’][1,]
$\tilde{\sigma}_{\varphi_{a_r}^{i}}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ phi0’][2,]
$\tilde{\theta}_{a_r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ theta0’][1,]
$\tilde{\sigma}_{\theta_{a_r}^{i}}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ spatialA0’][‘ theta0’][2,]
$\tilde{W}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ subdiag0’][‘ west0’][1,]
$\tilde{\sigma}_{W_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ subdiag0’][‘ west0’][2,]
$\tilde{E}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ subdiag0’][‘ east0’][1,]
$\tilde{\sigma}_{E_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘ AR0’][[r]][‘ subdiag0’][‘ east0’][2,]

$\tilde{N}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘north0’][1,]
$\tilde{\sigma}_{N_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘north0’][2,]
$\tilde{S}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘south0’][1,]
$\tilde{\sigma}_{S_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘south0’][2,]
$\widetilde{SE}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘southeast0’][1,]
$\tilde{\sigma}_{SE_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘southeast0’][2,]
$\widetilde{NW}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘northwest0’][1,]
$\tilde{\sigma}_{NW_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘northwest0’][2,]
$\widetilde{SW}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘southwest0’][1,]
$\tilde{\sigma}_{SW_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘southwest0’][2,]
$\widetilde{NE}_{0r}^{i}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘northeast0’][1,]
$\tilde{\sigma}_{NE_r^i}^{2}$	Object[‘Hyperpriors’][‘Xt0’][‘AR0’][[r]][‘subdiag0’][‘northeast0’][2,]

Figure 3.2: Hyperpriors model transcription to an object of class `HYPERPRIOR`

4. Example

The HBSTM package contains a real dataset in order to use it as example. The data store the temperature, which is collected in a grid of 70 points (n=7 x m=10 points in the grid) in the area that extends from 4º 30" W to 6º 30"W longitude, and from 35º 3"N to 36º 5"N.

The analyzed period covers January 1st 2009 to December 31st 2010; the frequency of the data is every 3 hours (temporal reference system is UTC); it starts at 00:00 (daily analysis) and forecasting is at 3:00, 6:00, 9:00, 12:00, 15:00, 18:00 and 21:00. The temperature is recorded every day, eight times a day; so we have a time series for each variable: one for each point on the grid with 5840 time observations.

Our purpose is to fit the data 'hirlam' using a HBSTM with the following characteristics:

- We fit the same spatial points. Then $\boldsymbol{K}$ is the identity matrix.
- Two seasonalities in $\vec{\boldsymbol{M}}_t$: $w_1 = 2\pi/2920$ and $w_2 = 2\pi/(2920/2)$.
- The autoregressive component $\vec{\boldsymbol{X}}_t$ is defined as:
$$\vec{\boldsymbol{X}}_t = \boldsymbol{H}_1\vec{\boldsymbol{X}}_{t-1} + \boldsymbol{H}_2\vec{\boldsymbol{X}}_{t-2} + \boldsymbol{H}_8\vec{\boldsymbol{X}}_{t-8} + \boldsymbol{H}_{16}\vec{\boldsymbol{X}}_{t-16} + \vec{\eta}_t$$
- The spatial structure of the model is five-lagged rose diagram.

Then, we loading the package HBSTM, the dataset 'hirlam' and its coordinates.

<u>R code and outputs:</u>

```
> set.seed(198193696)
> library(HBSTM)
> data(hirlam)
> dim(hirlam)
[1]   70 5840
> data(coordinates)
> dim(coordinates)
[1] 70  2
```

In the next sections we present the functions of the package by two points of views: A beginner user and an expert user.

4.1. Beginner user

In this section we present the main functions of HBSTM from a beginner point of view. In other words, we use the functions using the most common attributes to can fit the data and analyze the fitted model.

We have decided that our model fits the observed spatial points. Then, the $M \times S$ matrix $\boldsymbol{K}$ is the identity where $M = S$. The autoregressive temporal lags are 1, 8 and 16. The seasonal component is defined with an annual (2920) and semiannual (2920/2) components. Finally, we consider five spatial lags in each direction at the spatial component.

Although this methodology require thousands of iterations to fit the data, we fit the model only 10 iterations in order to get faster results.

R code and outputs:

```
> S=dim(coordinates)[1]
>model=hbstm(Zt=hirlam,K=diag(rep(1,S)),newGrid=coordinates,reglag=c(1,8,16
),seas=c(2920,2920/2),spatlags=c(5,5,5,5),nIter=10,fit=TRUE,save="Parameter
s")
-------- The approximated execution time is: 0.03269444 hours------
Iteration 1: MSE = 78530973
Iteration 2: MSE = 42624482
Iteration 3: MSE = 20338125
Iteration 4: MSE = 10366690
Iteration 5: MSE = 5812885
Iteration 6: MSE = 3650983
Iteration 7: MSE = 2474099
Iteration 8: MSE = 1839887
Iteration 9: MSE = 1409945
Iteration 10: MSE = 1235579
```

For each iteration, the function shows a plot with four graphics (Figure 4.1) where displays: the MSE of the execution; the observed values versus the predicted in a spatial point; and the ACF/PACF of the residuals in a spatial point.

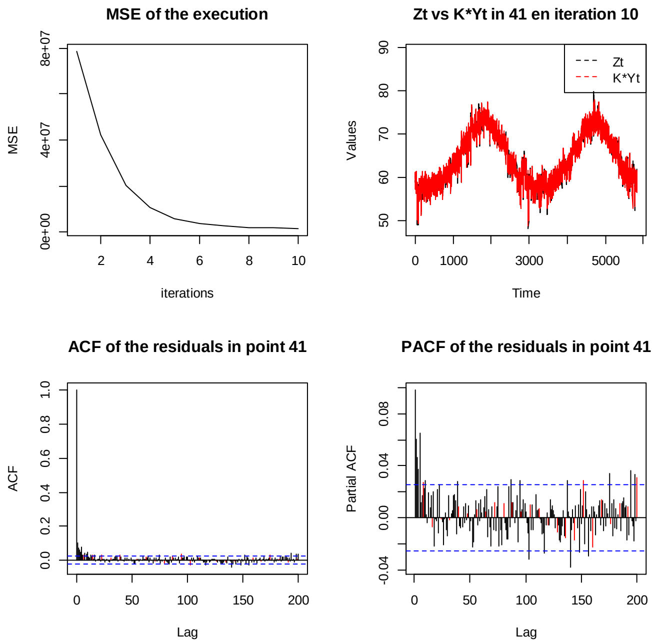


Figure 4.1: `hbstm` graphical output

Once the model is fitted, we proceed to check the results. The HBSTM provides different functions to analyze the residuals of the fitted model, the estimation and the MCMC samples of the parameters.

The first step is to study the model residuals. The object of class HBSTM `model` contains the residuals, then, to extract it we use `model[“residuals”]` and, for example, draw an histogram of a spatial point (Figure 4.2). Moreover, the function `plotRes` performs the figures 4.3 (a) and (b) in order to get more detailed residuals analysis and validate the model.

R code and outputs:

```
> hist(model["residuals"][1,],main="Histogram of the model residuals in the
spatial point 1",xlab="Residuals")
> plotRes(model)
```

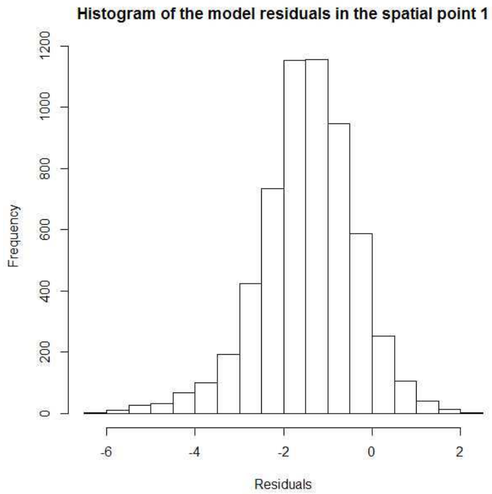
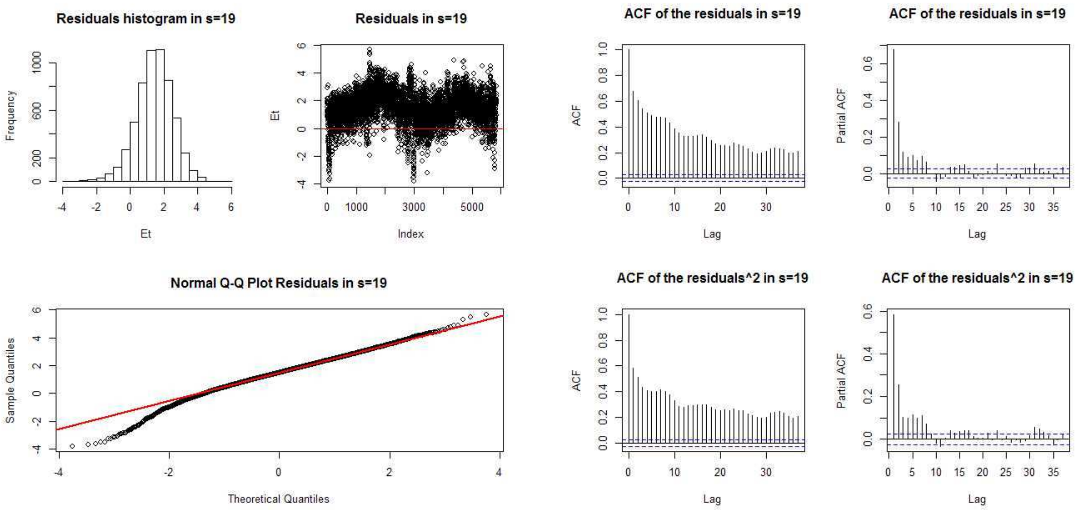


Figure 4.2: Histogram of the residuals in the spatial point 1



(a) (b)

Figure 4.3: `plotRes` output showing the residual analisis

To compare the spatial behavior in a specific time, we can use the function `plotFit` where shows a plot with the real $\vec{Z}_t$ observations, the $\vec{Z}_t$ estimations and the residuals in a fixed time (Figure 4.3). With this function we can easily see similarities or differences between the observed and estimated values. In our case the residuals are very bad because we have executed only 10 iterations.

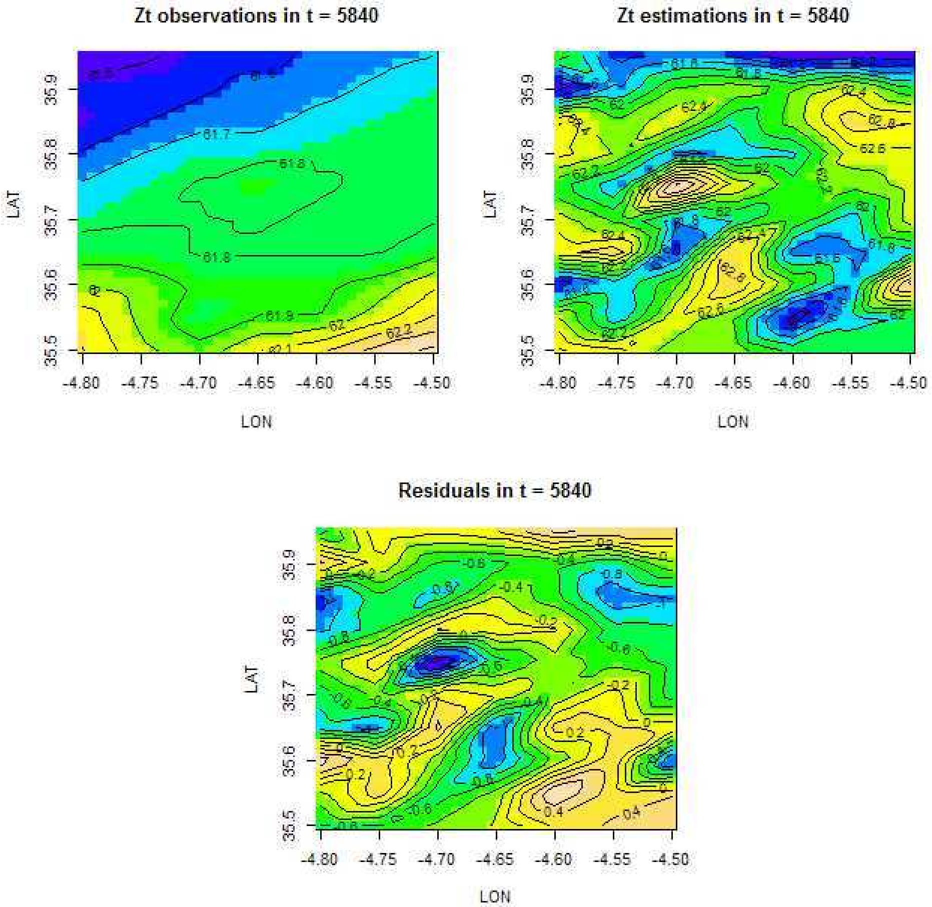


Figure 4.3: `plotFit` *output with the spatial fit analysis in the last temporal observation*

<u>R code and outputs:</u>

```
> plotFit(model)
```

Finally, we can study the values of the fitted model parameters. The function `results` shows, in the console and with plots, a summary containing the median and the 95% credibility intervals of the MCMC samples of the parameters. As example, in Figure 4.4 we can see the estimation of the seasonal parameters of $w_1 = 2\pi/2920$ (a) and the spatial parameters of $\vec{\boldsymbol{\mu}}$ (b).

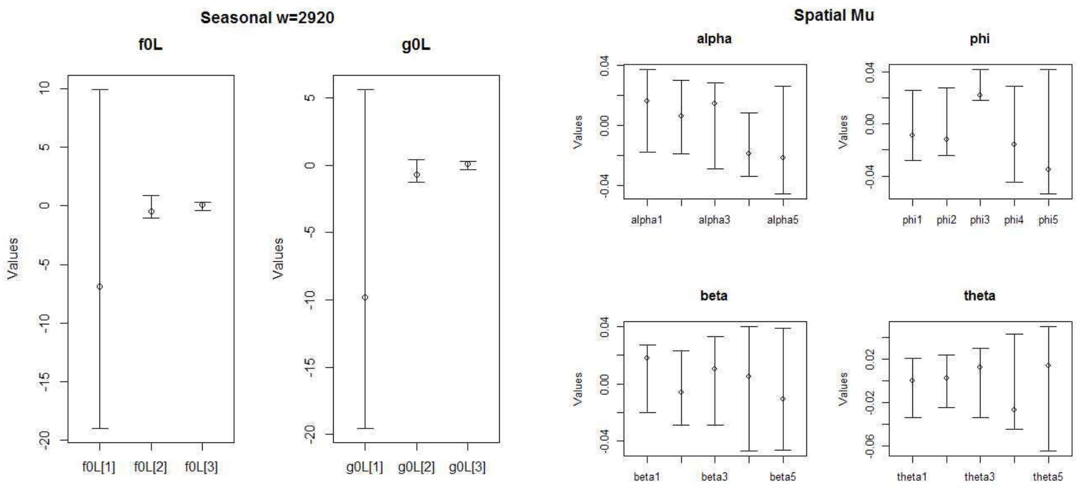


Figure 4.4: Part of the `results` output where shows the median and 95% credibility intervals for (a) the seasonal parameters in w=2920 and (b) the spatial parameters of $\vec{\boldsymbol{\mu}}$

<u>R code and outputs:</u>

```
> results(model,plots=TRUE)

…

------------------------------------------
              Mu component
------------------------------------------
…

---- Spatial parameters:
      lag1                     lag2                        lag3
alpha "0.007[-0.019,0.022]" "0.014[-0.022,0.024]"    "0.012[-0.017,0.028]"
beta  "0.014[-0.031,0.025]" "0.011[-0.032,0.036]"    "-0.021[-0.04,0]"
phi   "0.008[-0.029,0.022]" "-0.014[-0.034,-0.012]"  "0.024[-0.032,0.037]"
theta "0.023[-0.038,0.035]" "-0.018[-0.033,0.014]"   "-0.012[-0.041,0.027]"
      lag4                    lag5
alpha "0.009[-0.018,0.03]"    "-0.005[-0.027,0.017]"
beta  "-0.006[-0.042,0.026]"  "-0.003[-0.04,0.034]"
phi   "-0.028[-0.051,0.024]"  "0.021[-0.047,0.047]"
theta "0.01[-0.036,0.03]"     "-0.021[-0.043,0.039]"

------------------------------------------
              Mt component
------------------------------------------

----- Seasonal 1 (w=2920):
        Median  Low CI High CI
fvect   -4.630      NA      NA
f0L[1]  -9.859  -9.859  -9.859
f0L[2] -14.502 -14.502 -14.502
f0L[3] -18.069 -18.069 -18.069
gvect   -4.014      NA      NA
g0L[1]  -9.770  -9.770  -9.770
g0L[2] -17.614 -17.614 -17.614
g0L[3] -20.182 -20.182 -20.182
a0L[2]  6.841  6.841   6.841
a0L[3]  7.958  7.958   7.958
…
```

On the other side, in case we want to work with the median estimation of the parameters, the function `estimation` returns an object of class Parameters containing all the parameters estimations.

<u>R code and outputs:</u>

```
> est=estimation(model)
> est

-------------- Model definition ----------------
-- Seasonalities:
   w1   w2
 2920 1460
-- Autoregressive temp. lags:
    AR1 AR2 AR3
lag   1   8  16
-- Spatial lags:
 east-weast north-south southeast-northwest southwest-northeast
          5           5                   5                   5

--------- Values  of the Parameters ----------
---- Sigmas:
 sigma2E sigma2Y sigma2Mu sigma2A-1 sigma2A-8 sigma2A-16
   2.258   1.532    0.001     0.007     0.007      0.007

---- Mu component:
 muvect mean mu0vect mean mu0L[1] mu0L[2] mu0L[3]
    64.04633     64.04506  92.951   0.145  -0.787
-- Spatial Mu parameters:
        lag1   lag2  lag3   lag4   lag5
alpha  0.016  0.006 0.014 -0.019 -0.022
beta   0.018 -0.006 0.010  0.005 -0.011
phi   -0.009 -0.012 0.022 -0.016 -0.035
theta  0.000  0.002 0.012 -0.027  0.014

---- Mt component:
-- Seasonal w1 = 2920:
 fvect mean f0L[1] f0L[2] f0L[3] gvect mean g0L[1] g0L[2] g0L[3]
  -4.710886 -6.889 -0.454  0.095  -4.092614 -9.817 -0.706  0.148
-- Seasonal w2 = 1460:
 fvect mean f0L[1] f0L[2] f0L[3] gvect mean g0L[1] g0L[2] g0L[3]
 -0.1623429  9.564 -0.059 -0.282  0.9111714  0.411  0.244  0.039

---- Xt component:
-- Autoregressive 1 (t-1)
 avect mean a0vect mean a0L[1] a0L[2] a0L[3]
 0.06941429  0.08205714   2.92 -0.026 -0.011
-- Spatial avect parameters:
        lag1   lag2   lag3   lag4   lag5
alpha -0.009  0.002 -0.003  0.000 -0.006
beta   0.007 -0.013  0.011 -0.016 -0.025
phi   -0.014 -0.019 -0.005 -0.016 -0.019
theta -0.005  0.021 -0.016 -0.012  0.000
```

```
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.006  0.004  0.003  0.001  0.007
west      -0.007 -0.004 -0.002  0.002  0.010
north      0.007  0.013 -0.005 -0.003 -0.006
south      0.032  0.019  0.016  0.025  0.026
southeast  0.014  0.014  0.036  0.036  0.034
northwest -0.009  0.009 -0.002  0.011  0.005
southwest -0.003 -0.029  0.007  0.030  0.043
northeast -0.043 -0.032 -0.040 -0.020 -0.029
-- Autoregressive 2 (t-8)
  avect mean  a0vect mean a0L[1] a0L[2] a0L[3]
 -0.01424286 -0.001314286   2.92 -0.026 -0.011
-- Spatial avect parameters:
        lag1   lag2   lag3   lag4   lag5
alpha -0.009  0.002 -0.003  0.000 -0.006
beta   0.007 -0.013  0.011 -0.016 -0.025
phi   -0.014 -0.019 -0.005 -0.016 -0.019
theta -0.005  0.021 -0.016 -0.012  0.000
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.006  0.004  0.003  0.001  0.007
west      -0.007 -0.004 -0.002  0.002  0.010
north      0.007  0.013 -0.005 -0.003 -0.006
south      0.032  0.019  0.016  0.025  0.026
southeast  0.014  0.014  0.036  0.036  0.034
northwest -0.009  0.009 -0.002  0.011  0.005
southwest -0.003 -0.029  0.007  0.030  0.043
northeast -0.043 -0.032 -0.040 -0.020 -0.029

-- Autoregressive 3 (t-16)
  avect mean a0vect mean a0L[1] a0L[2] a0L[3]
 0.007714286  0.01075714   2.92 -0.026 -0.011
-- Spatial avect parameters:
        lag1   lag2   lag3   lag4   lag5
alpha -0.009  0.002 -0.003  0.000 -0.006
beta   0.007 -0.013  0.011 -0.016 -0.025
phi   -0.014 -0.019 -0.005 -0.016 -0.019
theta -0.005  0.021 -0.016 -0.012  0.000
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.006  0.004  0.003  0.001  0.007
west      -0.007 -0.004 -0.002  0.002  0.010
north      0.007  0.013 -0.005 -0.003 -0.006
south      0.032  0.019  0.016  0.025  0.026
southeast  0.014  0.014  0.036  0.036  0.034
northwest -0.009  0.009 -0.002  0.011  0.005
southwest -0.003 -0.029  0.007  0.030  0.043
northeast -0.043 -0.032 -0.040 -0.020 -0.029
```

4.2. Expert user

In this section we present the main functions of HBSTM from an expert point of view. Then, we use all the functions with all their features to fit the data and analyze the fitted model.

Again, we fit the data `hirlam` and we fit the same spatial grid. Then, the $M \times S$ matrix $\boldsymbol{K}$ is the identity where $M = S$. The function `hbstm` assign automatically the priors/hiperpriors and the initial values of the model parameters. This feature helps to the user avoiding to insert all the values before execute the MCMC algorithm. On the other hand, the automatically generated values has several contraries that we have to take into account:

- Being automatic values do not reflects the previous knowledge of the user.
- Could induce to converge to a local minimum.
- In high complex models, the values may not be good enough and the algorithm performance fails and crashes.

To avoid these tricky parts, we have implemented the option to not fit the model with the function `hbstm`. By this way, the function returns the object of class `HBSTM` containing all the model structure and all the automatic values before it is implemented. Then, we can define all the hiperpriors values and the initial values that we want to insert.

For example, fitting the `hirlam` model, we want to change the initial values of $\vec{\boldsymbol{\alpha}}_{\mu}$ and the hiperprior values of the σ_{η}^{2} parameter ($\tilde{q}_{\eta}$ and $\tilde{r}_{\eta}$). Then, we create the model with the `hbstm` function but assigning FALSE to the `fit` attribute. The next step is assign to the object `model` the desired values.

<u>R code and outputs:</u>

```
> S=dim(coordinates)[1]
>
model=hbstm(Zt=hirlam,K=diag(rep(1,S)),newGrid=coordinates,reglag=c(1,8,16)
,seas=c(2920,2920/2),spatlags=c(5,5,5,5),fit=FALSE)
>
> model["Parameters"]["Mu"]["spatialMu"]["alpha"]=c(0.2,0.1,-0.5,0,0.02)
>
> model["Hyperpriors"]["Xt0"]["sigmaN0"]=c(10,100)
```

Now, we use the function `hbstm.fit` to fit the modified model. To make faster the execution, again we execute the model only 10 iterations with a burning period of 2. Moreover, we indicate to the function to show the estimated time of execution and, for each iteration, the estimated time remaining execution and the execution plots. Also, we want that the function returns the mean and standard deviation of the fitted values and, finally, that returns the MCMC samples of the autoregressive temporal component ($\vec{\boldsymbol{X}}_t$) of the model.

R code and outputs:
```
>model=hbstm.fit(HBSTM=model,nIter=10,nBurn=2,time=TRUE,timerem=TRUE,plots=
TRUE,posterior="mean",save="Xt")
-------- The approximated execution time is: 0.03547222 hours------
Iteration 1: MSE = 78601235
Iteration 2: MSE = 42496634
-------- The approximated time remaining is: 0.02826667 hours------
Iteration 3: MSE = 20260160
-------- The approximated time remaining is: 0.023975 hours------
Iteration 4: MSE = 10354267
-------- The approximated time remaining is: 0.02231667 hours------
Iteration 5: MSE = 5859310
-------- The approximated time remaining is: 0.01926389 hours------
Iteration 6: MSE = 3632600
-------- The approximated time remaining is: 0.01558889 hours------
Iteration 7: MSE = 2451828
-------- The approximated time remaining is: 0.011875 hours------
Iteration 8: MSE = 1936936
-------- The approximated time remaining is: 0.008038889 hours------
Iteration 9: MSE = 1618933
-------- The approximated time remaining is: 0.00425 hours------
Iteration 10: MSE = 1512878
-------- The approximated time remaining is: 0 hours------
```

We have indicated that the model only save the MCMC sample of the autoregressive ($\vec{\boldsymbol{X}}_t$) component. This feature is very helpful because, fitting big datasets, the function requires less memory RAM. In case we want to study the rest of the parameters, we only have to fit more iterations with the function `hbstm.fit` and save the other structures.

Using the function `plotRes` we can check the quality of the residuals. In this case, we indicate to show the time residuals of the spatial point 1 and 500 lags with a period of 8 in the ACF/PACF plots. These results are shown in the figure 4.5. Again, the residuals shows a bad behavior because we have only executed the algorithm 10 iterations.

R code and outputs:
```
>plotRes(object=model,point=1,ARlags=500,ARperiod=8)
```

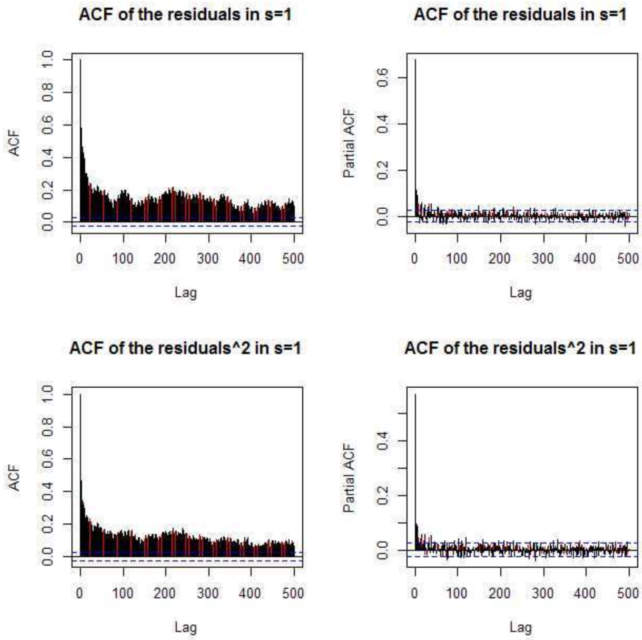


Figure 4.5: `plotRes` *output of the spatial point 1 residuals and squared residuals ACF/PACF showing the data period and 500 lags.*

In the next step, we check the parameters estimation using the function `results`. Since we have save the MCMC samples of $\vec{X}_t$, the function only shows the estimation of this part. Now, we want to check the estimation in the spatial point s=1 in the time t=1 and in the spatial point s=34 when t=2546 (Figure 4.6). Moreover, we change the % of the credibility intervals to 99 and the number of decimal digits is 2. We only show a part of the results.

R code and outputs:

```
>results(object=model,spatTemp=list(c(1,1),c(34,2546)),inter=0.99,digits=2,component="Xt",plots=TRUE
)

===================================================================
       Median and CI estimation of the model parameters
===================================================================

------------------------------------------
              Xt component
------------------------------------------
----- Xt spatio-temporal parameters:
                       Median Low CI High CI
Xt[(-4.8,35.95),1]      -0.07  -0.28    0.12
Xt[(-4.55,35.75),2546]  -0.62  -1.89    0.62

---------- Autoregressive 1 (t-1) ----------
----- Ar parameters:
        Median Low CI High CI
avect    0.078     NA      NA
a0vect   0.070     NA      NA
a0L[1]   2.390   2.39    2.39
a0L[2]   4.780   4.78    4.78
a0L[3]   9.830   9.83    9.83
sigma2A  0.010   0.00    0.05

---- Spatial parameters:
      lag1                 lag2                 lag3
alpha "0[-0.02,0.03]"      "0[-0.02,0.03]"      "-0.01[-0.02,0.02]"
beta  "0[-0.01,0.02]"      "-0.01[-0.03,0.01]" "0[-0.02,0.03]"
phi   "-0.01[-0.02,0.01]" "-0.01[-0.03,0.01]" "0[-0.04,0.03]"
theta "-0.01[-0.02,0.01]" "0[-0.03,0.03]"      "-0.01[-0.03,0.03]"
      lag4                 lag5
alpha "0.01[-0.03,0.02]"  "0[-0.02,0.02]"
beta  "-0.01[-0.03,0.02]" "-0.01[-0.04,0.01]"
phi   "-0.02[-0.04,0.05]" "-0.02[-0.05,0.05]"
theta "0[-0.03,0.04]"      "0[-0.04,0.03]"

---- Subdiagonal parameters of H matrix:
          lag1                 lag2                 lag3
east      "0.03[-0.03,0.08]"  "0.02[0,0.06]"       "0.02[0,0.06]"
west      "0.01[-0.03,0.08]"  "0.01[-0.01,0.07]"  "0.02[-0.02,0.05]"
north     "0.02[-0.01,0.06]"  "0.03[0,0.05]"       "0.02[0,0.05]"
south     "0.06[-0.02,0.1]"   "0.05[0,0.09]"       "0.03[0.01,0.08]"
southeast "0.04[-0.06,0.13]"  "0.01[-0.03,0.14]"  "0.03[-0.04,0.1]"
northwest "-0.01[-0.07,0.18]" "0.01[-0.04,0.14]"  "0.02[-0.02,0.16]"
southwest "0.02[-0.18,0.19]"  "0.03[-0.15,0.2]"   "0.01[-0.1,0.17]"
northeast "-0.01[-0.18,0.16]" "-0.01[-0.13,0.13]" "-0.01[-0.08,0.11]"
          lag4                 lag5
east      "0.02[-0.01,0.06]"  "0.02[-0.01,0.06]"
west      "0.02[-0.01,0.05]"  "0.03[-0.03,0.06]"
north     "0.02[0,0.05]"       "0.02[0,0.05]"
south     "0.04[0.01,0.08]"   "0.03[0.02,0.08]"
southeast "0.04[-0.04,0.09]"  "0.04[0,0.08]"
northwest "0.02[-0.03,0.17]"  "0.01[-0.08,0.16]"
southwest "0.03[-0.11,0.13]"  "0.05[-0.13,0.12]"
northeast "-0.03[-0.09,0.15]" "-0.04[-0.13,0.22]"
…
```

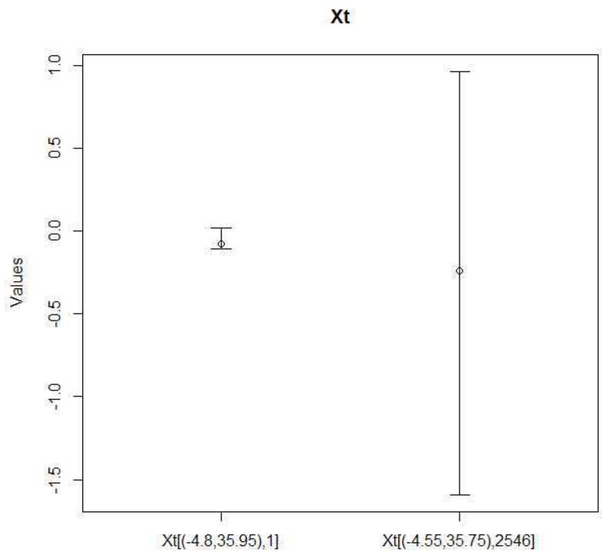


Figure 4.6: `results` *graphical output for* $\vec{\boldsymbol{X}}_t$ *median and 99% credibility interval estimation for the spatial point s=1 in the time t=1 and for the spatial point s=34 when t=2546*

`results` has other attribute called 'file'. It contains the name of a .tex file where all the output of `results` (the summary and the plots) will be stored.

Finally, in case we want the median estimated values of our MCMC parameters samples, we can use the function `estimate`. Moreover, we can decide the maximum number of digits the decimal numbers should have. In our case, we only save the MCMC samples of $\vec{\boldsymbol{X}}_t$, then, the rest of the parameters values are -9999999.

R code and outputs:

```
>results(object=model,spatTemp=list(c(1,1),c(34,2546)),inter=0.99,digits=2,
component="Xt",plots=TRUE,file="hirlamRep")
> estim=estimation(model,digits=3)
> estim

-------------- Model definition ----------------
-- Seasonalities:
   w1   w2
 2920 1460
-- Autoregressive temp. lags:
    AR1 AR2 AR3
lag   1   8  16
-- Spatial lags:
 east-weast north-south southeast-northwest southwest-northeast
          5           5                   5                   5

--------- Values  of the Parameters ----------
---- Sigmas:
  sigma2E  sigma2Y sigma2Mu sigma2A-1 sigma2A-8 sigma2A-16
 -9999999 -9999999 -9999999      0.01     0.008      0.009

---- Mu component:
 muvect mean mu0vect mean  mu0L[1]  mu0L[2]  mu0L[3]
    -9999999     -9999999 -9999999 -9999999 -9999999
-- Spatial Mu parameters:
          lag1     lag2     lag3     lag4     lag5
alpha -9999999 -9999999 -9999999 -9999999 -9999999
beta  -9999999 -9999999 -9999999 -9999999 -9999999
phi   -9999999 -9999999 -9999999 -9999999 -9999999
theta -9999999 -9999999 -9999999 -9999999 -9999999

---- Mt component:
-- Seasonal w1 = 2920:
 fvect mean   f0L[1]   f0L[2]   f0L[3] gvect mean   g0L[1]   g0L[2]
g0L[3]
   -9999999 -9999999 -9999999 -9999999   -9999999 -9999999 -9999999 -
9999999
-- Seasonal w2 = 1460:
 fvect mean   f0L[1]   f0L[2]   f0L[3] gvect mean   g0L[1]   g0L[2]
g0L[3]
   -9999999 -9999999 -9999999 -9999999   -9999999 -9999999 -9999999 -
9999999

---- Xt component:
-- Autoregressive 1 (t-1)
 avect mean a0vect mean a0L[1] a0L[2] a0L[3]
 0.07692857  0.07162857  4.249 -0.212 -0.104
-- Spatial avect parameters:
       lag1   lag2   lag3   lag4   lag5
alpha 0.008 -0.008 -0.001  0.001 -0.004
beta  0.011 -0.004  0.002 -0.013  0.006
phi   0.007 -0.003  0.001 -0.015  0.012
theta 0.009  0.003  0.014 -0.012  0.027
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.005  0.000 -0.003 -0.001  0.004
west      -0.009 -0.006 -0.003  0.000  0.009
north     -0.006  0.005 -0.003 -0.004  0.002
south      0.038  0.030  0.024  0.030  0.025
southeast  0.022  0.005  0.022  0.024  0.024
northwest -0.025 -0.004 -0.011 -0.002 -0.013
southwest  0.007  0.020 -0.016  0.014  0.037
northeast -0.018 -0.020 -0.044 -0.037 -0.042
```

```
-- Autoregressive 2 (t-8)
    avect mean a0vect mean a0L[1] a0L[2] a0L[3]
 -0.0006428571 0.005142857  4.249 -0.212 -0.104
-- Spatial avect parameters:
       lag1   lag2   lag3   lag4   lag5
alpha 0.008 -0.008 -0.001  0.001 -0.004
beta  0.011 -0.004  0.002 -0.013  0.006
phi   0.007 -0.003  0.001 -0.015  0.012
theta 0.009  0.003  0.014 -0.012  0.027
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.005  0.000 -0.003 -0.001  0.004
west      -0.009 -0.006 -0.003  0.000  0.009
north     -0.006  0.005 -0.003 -0.004  0.002
south      0.038  0.030  0.024  0.030  0.025
southeast  0.022  0.005  0.022  0.024  0.024
northwest -0.025 -0.004 -0.011 -0.002 -0.013
southwest  0.007  0.020 -0.016  0.014  0.037
northeast -0.018 -0.020 -0.044 -0.037 -0.042

-- Autoregressive 3 (t-16)
 avect mean a0vect mean a0L[1] a0L[2] a0L[3]
 0.01341429  0.02097143  4.249 -0.212 -0.104
-- Spatial avect parameters:
       lag1   lag2   lag3   lag4   lag5
alpha 0.008 -0.008 -0.001  0.001 -0.004
beta  0.011 -0.004  0.002 -0.013  0.006
phi   0.007 -0.003  0.001 -0.015  0.012
theta 0.009  0.003  0.014 -0.012  0.027
-- Space-time parameters:
            lag1   lag2   lag3   lag4   lag5
east       0.005  0.000 -0.003 -0.001  0.004
west      -0.009 -0.006 -0.003  0.000  0.009
north     -0.006  0.005 -0.003 -0.004  0.002
south      0.038  0.030  0.024  0.030  0.025
southeast  0.022  0.005  0.022  0.024  0.024
northwest -0.025 -0.004 -0.011 -0.002 -0.013
southwest  0.007  0.020 -0.016  0.014  0.037
northeast -0.018 -0.020 -0.044 -0.037 -0.042
```